An Algebraic Proof of the Generic Oddness of Equilibria of Finite Games

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Introduction

This paper seeks to provide a concise, modern proof of Wilson's classical oddness theorem (see [8]). It has been suggested in recent work that the generic finiteness and oddness of the equilibrium set for finite games can nowadays viewed as a direct corollary of the Kohlberg & Mertens structure theorem for the equilibrium manifold (e.g. [3], [7] Section I.4 Ex. 4b). To the extent of my knowledge, however, a proof of this implication appears nowhere in the published literature. As such, the purpose of this note is to provide such a proof.

Preliminaries

Game Theoretic

Let N be a finite set of agents. For all $n \in N$ let A_n denote agent n's finite action set and $A = \times_{n \in N} A_n$. Let Σ_n denote the simplex of probability measures on A_n, and $\Sigma = \times_{n \in N} \Sigma_i$, the $\sum_n (|A_n| - 1)$-dimensional cell of mixed strategies. Denote by $\Gamma_n = \mathbb{R}^{|S|}$ the set of potential utility functions for agent n, and define $\Gamma = \times_{n \in N} \Gamma_n$ as the space of potential payoff functions for of all players, identified with the space of games with agent set N and actions A. Let $\eta : \Gamma \rightrightarrows \Sigma$ denote the equilibrium correspondence, that is $G \mapsto \{\sigma \in \Sigma : \sigma \text{ is a Nash equilibrium of } G\}$, and let $E = \text{Graph}(\eta)$. Let $p : E \to \Gamma$ denote projection onto the first factor. Let $\bar{E}$ and $\bar{\Gamma}$ be the Alexandroff compactifications of E and Γ. For a map $t : E \to \Gamma$, let $\bar{t}$ denote the natural continuous extension of t to a map from $\bar{E} \to \bar{\Gamma}$ by letting $\bar{t}(\infty) = \infty$.

Semi-algebraic

Let X (resp. Y) be a semi-algebraic subset of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$), and let $f : X \to Y$ be a semi-algebraic map. The map f is said to be semi-algebraically trivial over a semi-algebraic subset $Z \subset \mathbb{R}^n$ if there is a semi-algebraic set W and a homeomorphism $h : f^{-1}(Z) \to Z \times W$ such that the following diagram commutes.

$$\begin{tikzcd}
f^{-1}(Z) \arrow[d, "f"'] \arrow[dr, "h"] & \\
Z & Z \times W \arrow[l, "\pi_1"]
\end{tikzcd}$$

The homeomorphism h is called a (semi-algebraic) trivialization of f over Z. A trivializing homeomorphism for f has the property that:

$$f(h^{-1}(z, w)) = z \qquad \forall z \in Z, w \in W.$$

We will make use of the following theorem due to Hardt (1980) (cf. [5], [2], [1]).

Theorem (Hardt, 1980)**.** *Let $f : X \to Y$ be a semi-algebraic map and let X (resp. Y) be a semi-algebraic subset of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$). Then there is a closed, strictly lower-dimensional semi-algebraic subset $Y_0 \subset Y$ such that for each of a finite number of open connected components Y_i of $Y \setminus Y_0$, there is a semi-algebraic trivialization of f over Y_i.*

The primary application of this result will be applied to the map $p : E \to \Gamma$. In [1] it is shown that η is a semi-algebraic correspondence, and E is a semi-algebraic set. As p is the projection map from E to Γ, it too is semi-algebraic and hence Hardt's theorem applies. For our purposes, the existence of a trivializing homeomorphism for p over some open ball Z about a game G guarantees a number of important properties. In particular that the set of equilibria of G is finite, the cardinality of the set of equilibria is constant over Z, and that the graph of the equilibrium correspondence over Z takes the form of a product of an open disk in Γ and some finite set.[1]

[1]It is worth noting that for the semi-algebraic set W_i that appears in the codomain of the the trivializing homeomorphism $q_i : p^{-1}(Y_i) \to Y_i \times W_i$ is necessarily zero dimensional and hence finite, as semi-algebraic sets have finitely many connected components. To see this, suppose it was not; then the trivialization gives a homeomorphism between $Y_i \times W_i$ and $p^{-1}(Y_i)$, and hence $\dim p^{-1}(Y_i) > \dim Y_i$ as dimension is a homeomorphism invariant (invariance of domain). But by the Kohlberg & Mertens homeomorphism, $\dim Y_i = \dim p^{-1}(Y_i)$, a contradiction.

The Structure Theorem

The structure theorem of Kohlberg & Mertens (1986) may be stated as follows.

Theorem (Theorem 1, Kohlberg & Mertens (1986))**.** *There exists a homeomorphism $\phi : E \to \Gamma$ such that $p \circ \phi^{-1}$ is (linearly) homotopic to the identity on Γ, under a homotopy that extends to $\bar{\Gamma}$.*

The Kohlberg & Mertens (1986) structure theorem was proven as part of a body of work seeking to formalize a game-theoretic notion of rationality that would exclude the normatively unreasonable predictions put forward by the classical Nash equilibrium concept or the multitude of other refinements. Kohlberg and Mertens sought, in particular, to prove that for every game, there was some *stable* equilibrium that was robust to measurement error in the payoffs, in essence that there was always some equilibrium for a given game such that all 'nearby' games had a 'nearby' equilibrium.

Kohlberg and Mertens' proof is constructive. They construct the homeomorphism ϕ by augmenting each action's probability in an equilibrium by it's expected payoff. This in turn is shown to be a reversible operation. A more in-depth discussion of the intuition underlying the construction of the homeomorphism ϕ is given in [4].

Main Results

We will be applying the traditional notion of Brouwer degree, which is defined for maps of spheres. Hence we will first verify that passing to compactifications has no effect on the relevant features of the maps defined above.

Proposition. *The extended map $\bar{\phi}$ is a homeomorphism.*

Proof. For locally compact Hausdorff spaces, it is a standard result that the natural extension is continuous if and only if ϕ is a proper map.[2][3] As ϕ is a homeomorphism,

[2] A map $f : X \to Y$ is proper if and only if, for all compacta $K \subseteq Y$, $\phi^{-1}(K)$ is compact.

[3] To see this, first suppose $f : X \to Y$ is proper and V is some open set in $\bar{Y}$. We must verify two cases, contingent upon whether or not ∞_Y is in V. If not, then $V \subseteq Y$ hence by continuity $f^{-1}(V)$ is open in $\bar{X}$. If, alternatively, $\infty_Y \in V$ then $Y \setminus V$ is compact. As f is presumed proper by hypothesis, it is the case that:

$$f^{-1}(Y \setminus V) = X \setminus f^{-1}(V \setminus \{\infty_Y\})$$

is a compact and therefore closed subset of X. Hence $\bar{f}^{-1}(V) = f^{-1}(V) \cup \{\infty_X\} = \bar{X} \setminus \big(X \setminus f^{-1}(V \setminus \{\infty_Y\})\big) = f^{-1}(V \setminus \{\infty_Y\})$ is open, and hence $\bar{f}$ is continuous. Alternatively, supposing that $\bar{f}$ is continuous, for all compact sets $K \subseteq Y$, $f^{-1}(A) = \bar{f}^{-1}(K)$ is a closed subset of a compact space, and hence compact.

it is trivially proper, hence $\bar{\phi}$ is continuous and indeed also a homeomorphism by an analogous argument applied to its inverse. $\square$

By the standard stereographic projection argument, $\bar{\Gamma}$ is a topological sphere of dimension $|N| \times |S|$, hence by the structure theorem (in particular the existence of ϕ), so too is $\bar{E}$. Since $\bar{p} \circ \bar{\phi}^{-1} : \bar{\Gamma} \to \bar{\Gamma}$ is a homeomorphism, its Brouwer degree is necessarily ± 1; since it is homotopic to the identity on $\bar{\Gamma}$, indeed $\deg(\bar{p} \circ \bar{\phi}^{-1}) = \deg(\mathbb{1}_{\bar{\Gamma}}) = 1$, as shown below.

Proposition. *The composition $\bar{p} \circ \bar{\phi}^{-1}$ is of Brouwer degree 1.*

Proof. This follows immediately from the homotopy invariance of Brouwer degree.[4] In particular, since homotopic maps induce the same map on homology, in light of the Kohlberg & Mertens (1986) structure theorem, we must have:

$$(\bar{p} \circ \bar{\phi}^{-1})_* = \mathbb{1}_{\bar{\Gamma}*} = \mathbb{1}_{\mathbb{Z}},$$

where the first equality stems from homotopy invariance. $\square$

We now turn to the proof our main result.

Theorem. *There is an open, dense set of games of full Lebesgue measure on which the set of Nash equilibria is both finite and odd.*

Proof. In light of the semi-algebraicity of p, by Hardt's theorem, there exists a closed, strictly lower dimensional set Γ_0 such that, for all finitely many open connected components Γ_i of $\Gamma \setminus \Gamma_0$, p is semi-algebraically trivial over Γ_i. Thus for all Γ_i there exists a trivializing homeomorphism $q_i : p^{-1}(\Gamma_i) \to \Gamma_i \times \{1, \ldots, K\}$ for some $K \in \mathbb{N}$.

Let $G \in \Gamma_i$, let $\varepsilon > 0$ such that $\mathrm{cl}(B_\varepsilon(G)) \subset \Gamma_i$ and define the restricted homeomorphism $\tilde{q} = q_i|_{p^{-1}(B_\varepsilon(G))} : p^{-1}(B_\varepsilon(G)) \to B_\varepsilon(G) \times \{1, \ldots, K\}$. As $\tilde{q}$ is a trivializing homeomorphism for p, $p \circ \tilde{q}^{-1}$ is bijective on $B_\varepsilon(G) \times \{k\}$ for all $k \in 1, \ldots, K$:

$$p(\tilde{q}^{-1}(G', k)) = G' \qquad \forall G' \in B_\varepsilon(G), \forall k \in 1, \ldots, K.$$

As a homeomorphism, $\tilde{q}^{-1}$ is an open map. Moreover, as η is lower hemi-continuous on $\Gamma \setminus \Gamma_0$, the projection from its graph to the domain, p, is also an open map. The composition of open maps is open, and thus $p \circ \tilde{q}^{-1}$ is an open bijection on each $B_\varepsilon(G) \times \{k\}$, and hence a homeomorphism.[5]

[4] This in turn follows from the homotopy invariance of the induced maps on homology.

[5] In particular, $|\eta(G')| = K$ for all $G \in B_\varepsilon(G)$. Given $\Gamma \setminus \Gamma_0$ is open, dense, and of full Lebesgue measure, we have also shown that the set of equilibria is generically finite.

Define the open sets $U_1, \dots, U_K$ by:

$$U_k = \phi \circ \tilde{q}^{-1}(B_\varepsilon(G) \times \{k\}) \qquad \forall k \in 1, \dots, K.$$

Then each U_k contains $G_k = \phi(\sigma_k^*) \in U_k$, for each $\sigma_k^* \in \eta(G)$, the game corresponding to the image under the Kohlberg & Mertens homeomorphism of each equilibrium of G. Then $p \circ \phi^{-1}$ takes each U_k homeomorphically onto $B_\varepsilon(G)$. This follows from the commutativity of the following diagram:

$$\begin{tikzcd}
B_\varepsilon(G) \times \{1, \dots, K\} \arrow[r, "\tilde{q}^{-1}"] \arrow[dr, "p \circ \tilde{q}^{-1}"'] & p^{-1}(B_\varepsilon(G)) \arrow[r, "\phi"] \arrow[d, "p"] & \amalg_k U_k \arrow[dl, "p \circ \phi^{-1}"] \\
 & B_\varepsilon(G) &
\end{tikzcd}$$

Formally, we have shown $\phi \circ \tilde{q}^{-1}$ is the composition of homeomorphisms and thus a homeomorphism, $p \circ \tilde{q}^{-1}$ was shown to be a homeomorphism on each $B_\varepsilon(G) \times \{k\}$, and $U_k = \phi \circ \tilde{q}^{-1}(B_\varepsilon(G) \times \{k\})$. As $p \circ \tilde{q}^{-1} = (p \circ \phi^{-1}) \circ (\phi \circ \tilde{q}^{-1})$, we then conclude $p \circ \phi^{-1}$ is indeed a homeomorphism, when restricted to each U_k.

Since $B_\varepsilon(G) \subset \Gamma = \bar{\Gamma} \setminus \{\infty_\Gamma\}$, the above also immediately holds for $\bar{p} \circ \bar{\phi}^{-1}$, the composite of the extensions, and we conclude it too acts homeomorphically on each U_k. Hence for all $k \in 1, \dots, K$:

$$\deg(\bar{p} \circ \bar{\phi}^{-1})_{G_k} = \pm 1,$$

where $\deg(\bar{p} \circ \bar{\phi}^{-1})_{G_k}$ is defined as the unique integer corresponding to the isomorphism of abelian groups:

$$(\bar{p} \circ \bar{\phi}^{-1})_* : H_{|N| \times |S|}(U_k, U_k \setminus \{G_k\}) \to H_{|N| \times |S|}(B_\varepsilon(G), B_\varepsilon(G) \setminus \{G\}).$$

By the local degree theorem (first equality), and in light of our above degree proposition (second equality),

$$\sum_{k=1}^{K} \deg(\bar{p} \circ \bar{\phi}^{-1})_{G_k} = \deg(\bar{p} \circ \bar{\phi}^{-1}) = 1,$$

so letting $n \in \mathbb{N}$ denote the number of G_k of local degree $+1$, $K = 2n - 1$, which is odd and finite for all $n \in \mathbb{N}$. As Γ_i was arbitrary, this holds on all of $\Gamma \setminus \Gamma_0$, albeit of course with differing K for each Γ_i. $\square$

Conclusions and Relation to Prior Proof Methods

This proof differs from prior methods in that the only game theoretic considerations are suppressed behind the Kohlberg & Mertens structure theorem. Wilson's original proof (see [8]) is essentially combinatorial in nature and involves varying single boundary conditions in agents' maximization problems holding others fixed and counting what amounts to paths through a graph. Harsanyi's 1973 proof (see [6]) instead makes use of smooth techniques tracing out the evolution solution set as agents' objective functions are continuously deformed from logarithmic to affine.

In comparison, the method of proof in the above is purely topological in nature, and makes no smoothness assumptions of any kind (indeed E is not a smooth manifold), relying only on invariants of maps between spheres.[6] Even the semi-algebraic tools of Hardt's theorem are required only to establish genericity: given Kohlberg & Mertens' structure theorem, at any game G for which there exists a $B_\varepsilon(G)$ such that p acts homeomorphically on each of the finitely many connected components of $\eta|_{B_\varepsilon(G)}$, the above oddness theorem would hold as a purely topological matter, with no further requirements, semi-algebraic or otherwise. In that sense, this method of proof lays bare that the oddness theorem is fundamentally mathematical, rather than game theoretic, in nature.

[6]Indeed the machinery employed above would hold equally well in general equilibrium contexts, though smooth tools there hold more applicability.

Appendix: Brouwer Degree

We first proceed with some elementary computations of homology groups of interest. In all that follows, $H_n(X)$ will denote reduced singular homology of the space X, with coefficients in $\mathbb{Z}$.

Proposition. *For all k, $H_k(S^n)$ is $\mathbb{Z}$ for $k = n$, and 0 for all other k.*

Proof. For $n > 0$ clearly $B^n \setminus \{0\}$ deformation retracts onto S^{n-1}, hence we consider the long exact sequence for the pair (B^n, S^{n-1}), identifying $H_k(B^n) \cong 0$ for all k by contractibility:

$$0 \longleftarrow H_{n-1}(S^{n-1}) \longleftarrow H_n(\underbrace{B^n/S^{n-1}}_{=S^n}) \longleftarrow 0$$

and hence the map $H_n(S^n) \to H_{n-1}(S^{n-1})$ is an isomorphism. The base case of $n = 0$ is satisfied immediately by definition of the reduced homology groups; proceeding inductively, in light of the above isomorphism yields the result. $\square$

Proposition. *For all $x \in S^n$, $H_n(S^n, S^n \setminus \{x\}) \cong \mathbb{Z}$.*

Proof. By the long exact sequence for the pair $(S^n, S^n \setminus \{x\})$, we get:

$$\cdots \longleftarrow H_{n-1}(S^n \setminus \{x\}) \xleftarrow{\delta} H_n(S^n, S^n \setminus \{x\}) \longleftarrow H_n(S^n) \longleftarrow H_n(S^n \setminus \{x\}) \longleftarrow \cdots$$

Identifying $H_n(S^n) \cong \mathbb{Z}$ and $H_k(S^n \setminus \{x\}) \cong 0$ for all $k \geq 1$ via contractibility, we get the short exact sequence:

$$0 \xleftarrow{\delta} H_n(S^n, S^n \setminus \{x\}) \longleftarrow \mathbb{Z} \longleftarrow 0$$

and hence $H_n(S^n, S^n \setminus \{x\}) \cong \mathbb{Z}$. $\square$

Now, let $f : S^n \to S^m$ be a continuous function, and $x \in S^n$. Then f induces a map:

$$f_* : H_n(S^n, S^n \setminus \{x\}) \to H_n(S^n, S^n \setminus \{f(x)\}).$$

In light of the above two propositions, this means $f_* : \mathbb{Z} \to \mathbb{Z}$, and hence f_* is multiplication by a constant, which we term the **local degree** of f and x, and denote by $\deg(f)_x$. Moreover, such a map also induces a homomorphism $f_* : H_n(S^n) \to H_n(S^n)$ for which we also have $f_* : \mathbb{Z} \to \mathbb{Z}$ and we refer to the integer multiplier as the **global degree** of f, or simply $\deg(f)$.

Theorem (Local Degree Theorem)**.** *Suppose $f : S^n \to S^n$ is a map. Suppose that there is some $y \in S^n$ such that:*

$$f^{-1}(y) = \{x_1, \dots, x_K\}$$

for $K < \infty$. Then:

$$\deg(f) = \sum_{k=1}^{K} \deg(f)_{x_k}$$

Proof. Note first that instead of computing the local degree at x_k via $H_n(S^n, S^n \setminus x_k)$, it suffices to instead pick a neighborhood U_k of x_k, and a neighborhood V of y such that $f(U_k) \subseteq V$ and then instead look at the map:

$$f_* : H_n(U_k, U_k \setminus \{x_k\}) \to H_n(V, V \setminus \{y\}).$$

This follows because the excision axiom for homology states that if (X, A) is a pair and B a subset of A such that the closure of B is contained in the interior of A, then the inclusion map $i : (X \setminus B, A \setminus B) \hookrightarrow (X, A)$ induces an isomorphism on homology, i_*, hence letting $A = S^n \setminus \{x_k\}$ and $B = U_k^c$ (and $A' = S^n \setminus \{y\}$, $B' = V^c$), the following diagram commutes:

$$\begin{tikzcd}
H_n(S^n, S^n \setminus \{x_k\}) \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{y\}) \\
H_n(U_k, U_k \setminus \{x_k\}) \arrow[r, "f_*"] \arrow[u, "{i_*, \cong}"] & H_n(V, V \setminus \{y\}). \arrow[u, "{i'_*, \cong}"']
\end{tikzcd}$$

Now, as a subset of $\mathbb{R}^{n+1}$, S^n is Hausdorff and hence we may without loss take the U_k to be disjoint. Consider the following commutative diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d] & H_n(S^n) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{x_1, \dots, x_K\}) \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{y\}) \\
H_n(\amalg_k U_k, \amalg_k (U_k \setminus \{x_k\})) \arrow[u, "{\text{excision}, \cong}"] & \\
\bigoplus_k H_n(U_k, U_k \setminus \{x_k\}) \arrow[r, "\oplus f_*"] \arrow[u, "\cong"] & H_n(V, V \setminus \{y\}) \arrow[uu, "\cong"']
\end{tikzcd}$$

where the horizontal maps are all induced by f or restrictions of f to suitable subspaces, the top-left vertical map is induced by the identity $\mathbb{1}_{S^n}$, the bottom-left is induced by the respective identities $\mathbb{1}_{U_k}$, and the middle-left vertical map is induced by the various inclusions $i_k : U_i \hookrightarrow S^n$.

Consider the generator $1 \in H_n(S^n)$. Traversing the left-hand side of the diagram, the image of this generator in $\bigoplus_k H_n(U_k, U_k \setminus \{x_k\})$ is $(1, \ldots, 1)$. The bottom map sends this to $\sum_k \deg(f)_{x_k}$. But traversing the upper and right sides of the diagram, the image of 1 is $\deg(f)$. Hence by commutativity we obtain our result. $\square$

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